The Detour Vertex Covering Number of a Graph

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ABSTRACT

A vertex subset S of a connected graph G of order at least two is called a detour vertex cover if S is both a detour set and a vertex cover. The least cardinality of a detour vertex cover is the detour vertex covering number of G denoted as $dn_\alpha(G)$. Any detour vertex cover of cardinality $dn_\alpha(G)$ is a dn_α-set of G. Some general properties satisfied by detour vertex covering number of a graph are studied. The detour vertex covering number of some standard graphs are determined. Some bounds for $dn_\alpha(G)$ are obtained and the graphs attaining these bounds are characterized.

Keywords: *Detour dominating set, Detour set, Detour vertex cover, Vertex cover*
AMS Subject Classification (2010): 05C12

1. Introduction

We refer to Harary [1] for basic definitions. We consider finite, undirected, connected graphs without loops and multiple edges. The number of vertices and edges of a graph G are denoted as $n = V(G)$ and $m = E(G)$ respectively. A vertex v is a simplicial vertex or an extreme vertex of G if the subgraph induced by its neighbours is complete. A vertex of degree 1 is called an end vertex. [2]

For vertices u and v in a connected graph G, the *detour distance* $D(u,v)$ is the length of a longest $u-v$ path in G. A $u-v$ path of length $D(u,v)$ is called a $u-v$ detour. These concepts were studied by Chartrand [3,4]. A vertex x is said to lie on a $u-v$ detour P if x is a vertex of a $u-v$ detour path P including the vertices u and v. A set $S \subseteq V(G)$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S. The $detour\ number$ $dn(G)$ is called a minimum order of a detour set and any detour set of order $dn(G)$ is called a minimum detour set of G. These concepts were studied by Chartrand [5].

A subset $S \subseteq V(G)$ is called a *vertex covering set* or *vertex cover* of G if every edge has at least one end vertex in S. A vertex covering set with minimum cardinality is a *minimum vertex covering set* of G. The *vertex covering number* of G is the cardinality of any minimum vertex covering set of G denoted as $\alpha(G)$. The vertex covering number of a graph was studied in [6].

A set of vertices in a graph G is *independent* if no two of the vertices are adjacent. A subset $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. A *detour dominating set* of G is a subset S of vertices which is both a detour set and a dominating set. The minimum cardinality of a detour dominating set of a graph G is its *detour domination number*. The concept of detour dominating set was introduced by Chartrand [7] and further studied in [8].

In this paper, we define detour vertex covering number $g_\alpha(G)$ of a graph and initiate a study of this parameter. We investigate about some general properties satisfied and some bounds attained by this parameter. We need the following theorems.

Theorem 1.1 [9] Every end vertex of a connected graph G belongs to every detour set of G.

Theorem 1.2 [10] If G is a connected graph of order $n \geq 2$, then $g_\alpha(G) = 2$, if and only if G is either K_2 or $K_{2,n-2}(n \geq 3)$.

Theorem 1.3 [10] Let T be a tree of order $n \geq 2$. Then the following statements are equivalent.

(i) $g_\alpha(T) = g(T)$

(ii) T is a star.

(iii) $\alpha(T) = 1$

(iv) The set of all end vertices of T is a vertex cover of T.

Theorem 1.4. [10] Let T be a tree of order $n \geq 3$. Then $g_\alpha(T) = n - 1$ if and only if T is either a star or a double star.

2. Results and Discussion

Detour vertex covering number of a graph

Definition 2.1 A vertex subset S of a graph G is known as a detour vertex cover of G if it is both a detour set and a vertex cover of G. The least cardinality of a detour vertex cover is called the detour vertex covering number denoted as $dn_\alpha(G)$. Any detour vertex cover of G with cardinality $dn_\alpha(G)$ is called as a minimum detour vertex cover of G. A minimum detour vertex cover of G is also known as dn_α-set of G.

Example 2.2 In this graph G, we observe that the set $S = \{v_2, v_5\}$ is a minimum detour set but its vertices are not incidenting all the edges of G. Also note that $S_1 = \{v_2, v_3, v_4, v_5\}$ is both a detour set and a vertex cover of G. Hence it is a detour vertex cover of G. And there does not exist a detour vertex cover of cardinality 3. Hence S_1 is a dn_α-set of G. Thus $dn_\alpha(G) = 4$.

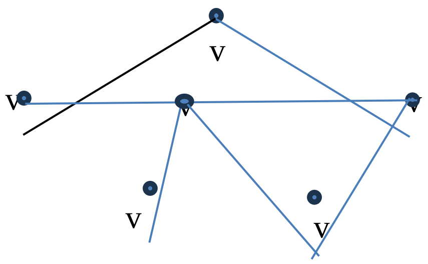


Figure 1

Remark 2.3 Graph in Figure 1 has $S = \{v_2, v_3, v_4\}$ as a minimum vertex cover but it is not a detour set. Thus, a vertex cover need not be a detour vertex cover of G. A graph G can allow more than one dn_α-set. For example, the graph G in Figure 1 has both $S_1 = \{v_2, v_3, v_4, v_5\}$ and $S_2 = \{v_2, v_4, v_5, v_6\}$ as dn_α-sets of G.

Theorem 2.4 Let G be any connected graph. Then $2 \leq \max\ \{\alpha\,(G), dn(G)\} \leq dn_\alpha(G \leq n$.

Proof. At least two vertices must belong to any detour set and so $2 \leq \max\ \{\alpha\,(G), dn(G)\}$. It follows from the definition of detour vertex covering number of G that a detour vertex cover must be both a detour set and a vertex cover of G. Thus we have $\max\ \{\alpha\,(G), dn(G)\} \leq dn_\alpha(G)$. Also $V(G)$ is itself a detour vertex cover of G. Hence we have $dn_\alpha(G) \leq n$. Thus $2 \leq \max\ \{\alpha\,(G), dn(G)\} \leq dn_\alpha(G) \leq n$.

Theorem 2.5 Each detour vertex cover of a connected graph Gcontains each end vertex of G.

Proof. Since every detour vertex cover is also a detour set of G, the proof follows from Theorem 1.1.

Corollary 2.6 Let $K_{1,n-1} (n \geq 3)$ be a star. Then $dn_\alpha\left(K_{1,n-1}\right) = n - 1$.

Proof. The result follows from Theorem 2.5.

Remark 2.7 It is not necessary that every extreme vertex of G belong to every detour vertex cover of G. For example, the graph G in Figure 2, has two extreme vertices namely v_2, v_3. Here the set $S_3 = \{v_2, v_3\}$ is a minimum detour set of G. But it does not cover the edge $v_1 v_4$. Thus S_3 is not a detour vertex covering set of G and the set $S_4 = \{v_1, v_4\}$ forms a detour vertex cover of G. Hence in this case no extreme vertex belongs to the detour vertex cover of G. Consider the graph G in Figure 3, it has three extreme vertices v_1, v_3, v_6. Of that v_6 is specifically an end vertex of G. It is easily observed that $S_5 = \{v_1, v_2, v_5, v_6\}$ is a dn_α-set of G. Here the end vertex v_6 of the graph belongs to the dn_α-set S_5 and the extreme vertex $v_1$belongs to the dn_α-set S_5 but the extreme vertex $v_3$does not belong to S_5.

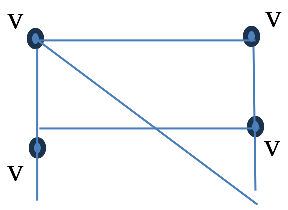
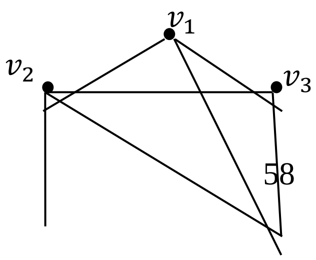


Figure 2



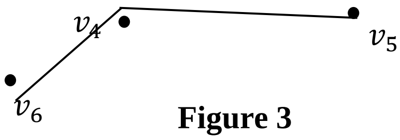


Figure 3

Theorem 2.8 If G is a connected graph of order $n \geq 2$, then

(i) $dn_\alpha(G) = 2$ if and only if G is either $K_{2,n-2}(n \geq 3)$ or there exists a hamiltonian path joining any two vertices u, v of G such that $V(G) - \{u, v\}$ is either empty or independent.

(ii)$dn_\alpha(G) = n$ if and only if $n = 2$. That is, $dn_\alpha(G) = n$ if and only if $G = K_2$.

Proof. (i) Let $dn_\alpha(G) = 2$. Let $S = \{u, v\}$. We show that either $G = K_{2,n-2}$ or there exists a hamiltonian path joining the vertices of S such that $V(G) - S$ is either empty or independent. Suppose that $G = K_{2,n-2}$. Since in a complete bipartite graph, every edge joins a vertex of one partite set to a vertex of the other partite set, the shortest path joining any two vertices of $K_{2,n-2}$, is also the longest path joining the two vertices. Hence by Theorem 1.2, $dn_\alpha(G) = g_\alpha(G) = 2$ and hence the proof holds. If not, assume that there exists a hamiltonian path joining any two vertices u, v of G. Then $S = \{u, v\}$ serves as a minimum detour set of G and so $dn(G) = 2$. Since $dn_\alpha(G) = 2$, it follows that each edge of G has at least one end vertex in S, and so no two vertices in $V(G) - S$ are joined by an edge in G. Thus $V(G) - S$ is either empty or independent.

Conversely, let G=$K_{2,n-2}(n \geq 3)$. Since in a complete bipartite graph, the shortest path joining any two vertices is also the longest path joining the two vertices, we have by Theorem 1.2, $g_\alpha(G) = dn_\alpha(G) = 2$. Suppose there exists a hamiltonian path joining any two vertices u, v of G such that $V(G) - \{u, v\}$ is either empty or independent. Let $S = \{u, v\}$. Then S is a minimum detour set of G. Since $V(G) - S$ is either empty or independent, no two vertices of $V(G) - S$ are adjacent in G. Thus $V(G) - S$ is also a vertex cover of G. Hence $V(G) - S$ is a minimum detour vertex cover of $G$and hence $dn_\alpha(G) = |S| = 2$.

(ii) Let $dn_\alpha(G) = n$. Suppose that the graph G has a path of length at least 3. Then there must exist at least one vertex of G which lies internally on this path and so $dn_\alpha(G) \leq n - 1$. Since, $dn_\alpha(G) = n, G$ must not admit a path of length 3 or more and since it is a connected graph, the only possibility is $G = K_2$.

Conversely, suppose that$G = K_2$. Then clearly, $dn_\alpha(G) = 2 = n$.

Theorem 2.9 Let G be a connected graph with $dn(G) \geq n - 1$. Then $dn_\alpha(G) = dn(G)$.

Proof. Let G be a connected graph with $dn(G) \geq n - 1$. By Theorem 2.4, $dn(G) \leq dn_\alpha(G) \leq n$. If $dn(G) = n$, then $dn_\alpha(G) = n$ and so $dn(G) = dn_\alpha(G)$. If $dn(G) = n -$

1, then let $S = \{x_1, x_2, \ldots, x_{n-1}\}$ be a minimum detour set of G. Let there exist a vertex say, x not in S. Then x lies on a detour P joining any two vertices of S. Then all the edges of G including the edges adjacent to x are covered by the vertices of S. Hence S is a dn_α-set of G. Thus $dn(G) = dn_\alpha(G)$.

Theorem 2.10 Let G be a connected graph of order $n > 2$. Then $dn_\alpha(G) = dn(G)$ if and only if there exists a minimum detour set S such that $V(G) - S$ is independent.

Proof. Suppose that $dn_\alpha(G) = dn(G)$. Let S be a minimum detour set of G. Since $dn_\alpha(G) = dn(G)$, S is a detour vertex cover of G. Then each edge of G is incident with at least one vertex in S and so no edge of G has two ends in $V(G) - S$. Thus no pair of vertices of $V(G) - S$ are adjacent and hence $V(G) - S$ is an independent set.

If S is a minimum detour set of G such that $V(G) - S$ is independent, then no pair of vertices of $V(G) - S$ are adjacent. Thus every edge of G has at least one end in S. Hence S is a minimum detour vertex cover of G so that $|S| = dn(G)$=$dn_\alpha(G)$.

Theorem 2.11 Let T be a tree of order $n \geq 2$. Then the following statements are equivalent.

(i) $dn_\alpha(T) = dn(T)$.

(ii) T is a star.

(iii) $\alpha(T) = 1$

(iv) The set of all end vertices of T is a vertex cover of T.

Proof. Since in a tree, there exists a unique path between every pair of vertices, the shortest path and the longest path connecting a pair of vertices are the same. Hence the result follows from Theorem 1.3.

Theorem 2.12 Let T be a tree of order $n \geq 3$. Then $dn_\alpha(T) = n - 1$ if and only if $T$is either a star or a double star.

Proof. Since in a tree, there exists a unique path between every pair of vertices, the result follows from Theorem 1.4.

Theorem 2.13 For the complete graph $K_n (n \geq 3)$, $dn_\alpha(K_n) = n - 1$.

Proof. Since in a complete graph, every pair of distinct vertices are adjacent, at least $n - 1$ vertices of K_n are needed to cover the all the edges of K_n. Also any pair of adjacent vertices of K_n forms a minimum detour set of K_n. Then by Theorem 2.4, $dn_\alpha(K_n) \geq n - 1$. Also by Theorem 2.8 (ii), $dn_\alpha(K_n) \neq n$, since $n \geq 3$. Thus $dn_\alpha(K_n) = n - 1$.

Theorem 2.14 For the wheel $W_n = K_1 + C_{n-1} (n \geq 5), dn_\alpha(W_n) = \left\lceil \frac{n-1}{2} \right\rceil + 1$.

Proof. Let $C_n: v_1, v_2, \dots, v_{n-1}, v_1$ be the cycle of W_n and x, the vertex of K_1 in W_n. Then $S = \left\{x, v_1, v_3, \dots, v_{2\left\lceil\frac{n-1}{2}\right\rceil-1}\right\}$ is a dn_α-set of W_n. Hence $dn_\alpha(W_n) = \left\lceil\frac{n-1}{2}\right\rceil + 1$.

Theorem 2.15 Let S be a minimum detour dominating set of a connected graph G. Then S is a detour vertex cover of G if and only if $V(G) - S$ is an independent set.

Proof. Let S be a minimum detour dominating set of G. If S is a detour vertex cover of G, then every edge of G has at least one end in S. Hence no two vertices in $V(G) - S$ are adjacent so that $V(G) - S$ is independent.

Conversely, let $V(G) - S$ be an independent set of G. Then every edge of G has at least one end in S so that S is also a vertex cover of G. Hence $S$is a detour vertex cover of G.

3. Conclusion

In this paper, we introduced a new graph theoretic parameter, ‘The detour vertex covering number’ of a graph and discussed its properties. Detour vertex covering number of some standard graphs like complete graph, star graph, wheel graph are determined. Connected graphs of order n with detour vertex covering number 2 and n are characterized. The results given in this paper will be useful in the future study on connected detour vertex covering number, upper detour vertex covering number, open detour vertex covering number etc.

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